Floating Points

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The Problem

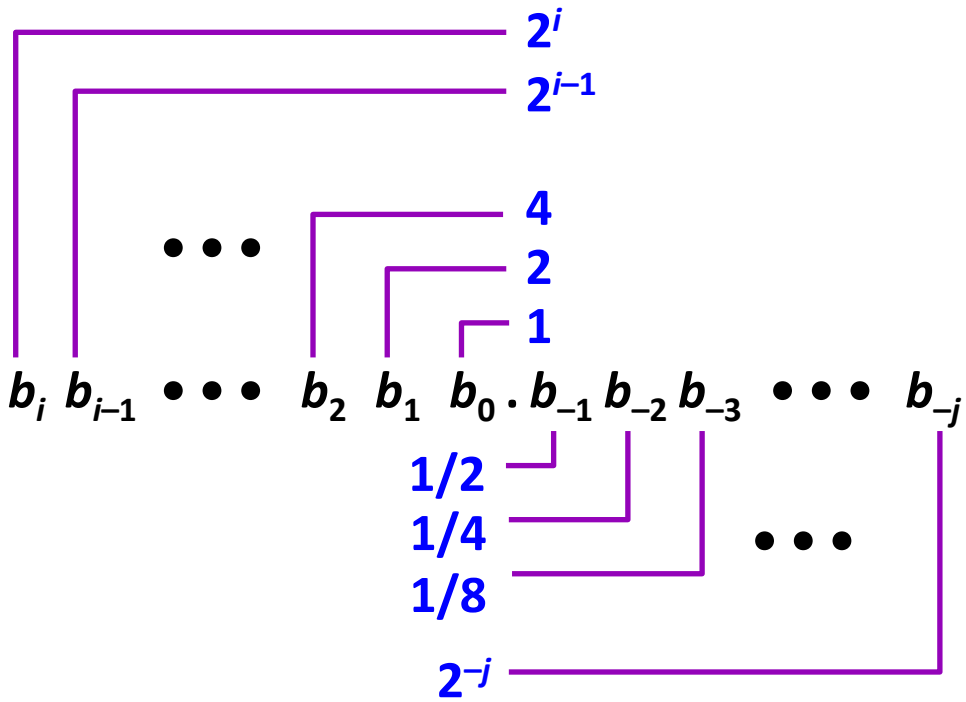
- How to represent fractional values with finite number of bits?
  - 0.1
  - 0.612
  - 3.14159265358979323846264338327950288...

- Wide ranges of numbers
  - 1 Light-Year = 9,460,730,472,580.8 km
  - The radius of a hydrogen atom: 0.000000000025 m

Fractional Binary Numbers (1)

- Representation
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number:

$$\sum_{k=-j}^{i} b_k \cdot 2^k$$



Fractional Binary Numbers (2)

- Examples:

Value	Representation
$5\ ^{3}/_{4}$	101.11_2
$2\ ^{7}/_{8}$	10.111_2
$^{63}/_{64}$	0.111111_2

- Observations
  - Divide by 2 by shifting right
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Fractional Binary Numbers (3)

- Representable numbers
  - Can only exactly represent numbers of the form x / 2^k
  - Other numbers have repeating bit representations

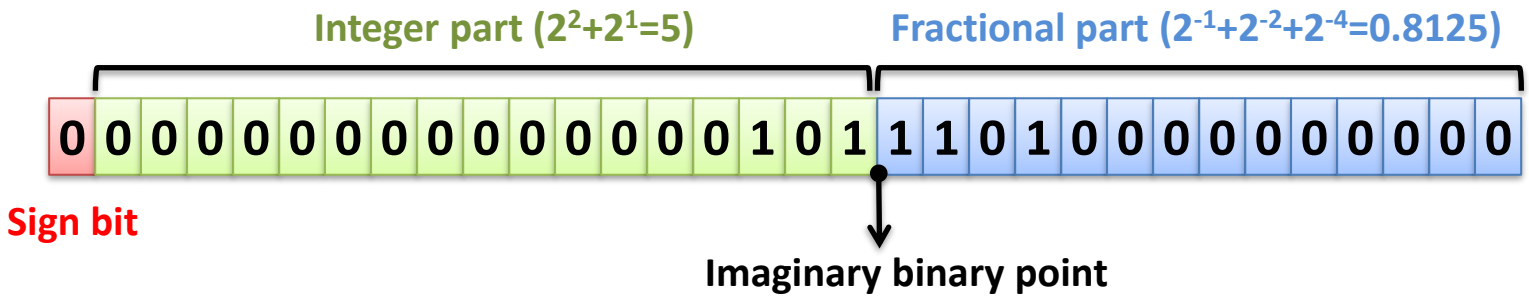
Value	Representation
$1/3$	$0.0101010101[01]\ldots_2$
$1/5$	$0.001100110011[0011]\ldots_2$
$1/10$	$0.0001100110011[0011]\ldots_2$

Fixed Points

Fixed-Point Representation (1)

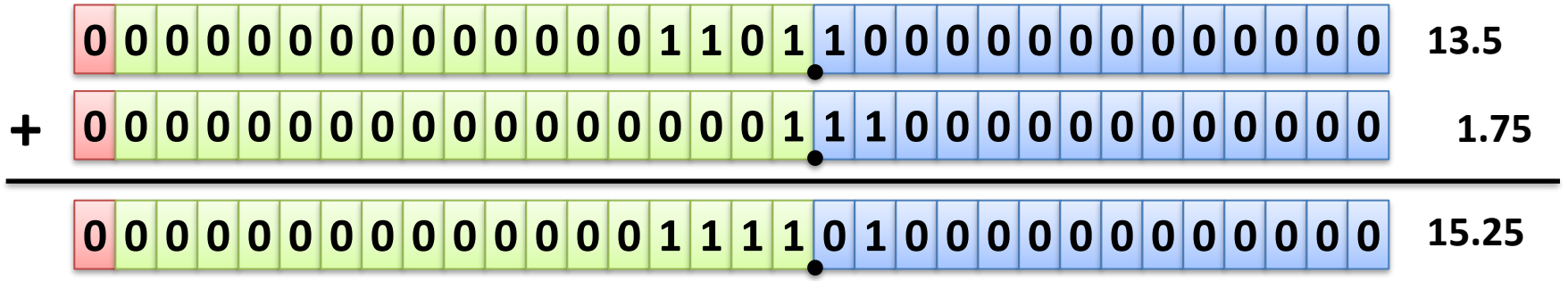
- ***p.q*** Fixed-point representation
  - Use the rightmost ***q*** bits of an integer as representing a fraction
  - Example: 17.14 fixed-point representation
    - 1 bit for sign bit
    - 17 bits for the integer part
    - 14 bits for the fractional part
    - An integer ***x*** represents the real number ***x*** / 2^{14}
    - Maximum value: $(2^{31} - 1) / 2^{14} \fallingdotseq 131071.999$

$$-b_p \cdot 2^p + \sum_{k=-q}^{p-1} b_k \cdot 2^k$$



Fixed-Point Representation (2)

- Properties
  - Convert ***n*** to fixed point: $n * f \quad (= n << q)$
  - Add ***x*** and ***y***: $x + y$



  - Subtract ***y*** from ***x***: $x - y$
  - Add ***x*** and ***n***: $x + n * f$
  - Multiply ***x*** by ***n***: $x * n$
  - Divide ***x*** by ***n***: x / n

***x, y*: fixed-point number**
***n*: integer**
f* = 1 << *q

Fixed-Point Representation (3)

- Pros
  - Simple
  - Can use integer arithmetic to manipulate
  - No floating-point hardware needed
  - Used in many low-cost embedded processors or DSPs (digital signal processors)

- Cons
  - Cannot represent wide ranges of numbers

Representing Floating Points

Chap. 3.5

Representing Floating Points

- IEEE standard 754
  - Established in 1985 as uniform standard for floating-point arithmetic
    - Before that, many idiosyncratic formats
    - Portability issues for scientific code
  - Supported by all major CPUs
  - William Kahan, a primary architect of IEEE 754, won the Turing Award in 1989
  - Driven by numerical concerns
    - Nice standards for rounding, overflow, underflow
    - Hard to make go fast
    - Numerical analysts predominated over hardware types in defining standard

FP Representation

- Numerical form: $-1^s \times M \times 2^E$
  - Sign bit ***s*** determines whether number is negative or positive
  - Significand ***M*** normally a fractional value in range [1.0, 2.0)
  - Exponent ***E*** weights value by power of two

- Encoding

  - MSB is sign bit ***s*** (0: non-negative, 1: negative)
  - ***exp*** field encodes ***E*** (Exponent)
  - ***frac*** field encodes ***M*** (Mantissa)

FP Precisions

- Single precision
  - 8 ***exp*** bits, 23 ***frac*** bits (32 bits total)
- Double precision
  - 11 ***exp*** bits, 52 ***frac*** bits (64 bits total)
- Extended precision
  - 15 ***exp*** bits, 63 ***frac*** bits
  - Only found in Intel-compatible machines
  - Stored in 80 bits (1 bit wasted)

Normalized Values

- Condition: ***exp*** ≠ ***000…0*** and ***exp*** ≠ ***111…1***
- Exponent coded as a biased value
  - ***E*** = ***Exp*** – ***Bias***
  - ***Exp***: unsigned value denoted by ***exp***
  - ***Bias***: Bias value (=$2^{k-1}-1$, where k is the number of ***exp*** bits)
    - Single precision (k=8): 127 (***Exp***: 1..254, ***E***: -126..127)
    - Double precision (k=11): 1023 (***Exp***: 1..2046, ***E***: -1022..1023)
- Significand coded with implied leading 1
  - ***M*** = $1.xxx\ldots x_2$
    - Minimum when ***frac*** = 000…0 (***M*** = 1.0)
    - Maximum when ***frac*** = 111…1 (***M*** = 2.0 – ε)
  - Get extra leading bit for "free"

Normalized Values: Example

- `float f = 2003.0;`
  - 2003_{10} = 11111010011_2 = $1.11111010011_2 \times 2^{10}$
- Significand
  - ***M*** = 1.11111010011_2
  - ***frac*** = $11111010011000000000000_2$
- Exponent
  - ***E*** = 10
  - ***Exp*** = ***E*** + ***Bias*** = 10 + 127 = 137 = 10001001_2

Hex:	**4**	**4**	**F**	**A**	**6**	**0**	**0**	**0**
Binary:	**0100**	**0100**	**1111**	**1010**	**0110**	**0000**	**0000**	**0000**
137:	**100**	**0100**	**1**					
2003:			***1***111	1010	0110			

Denormalized Values

- Condition: ***exp = 000…0***
- Value
  - Exponent value ***E*** = 1 – ***Bias***
  - Significand value ***M*** = 0.xxx…x_2 (no implied leading 1)
- *Case 1*: ***exp = 000…0, frac = 000…0***
  - Represents value 0.0
  - Note that there are distinct values +0 and -0
- *Case 2*: ***exp = 000…0, frac ≠ 000…0***
  - Numbers very close to 0.0
  - "Gradual underflow": possible numeric values are spaced evenly near 0.0

Special Values

- Condition: ***exp = 111…1***
- *Case 1*: ***exp = 111…1, frac = 000…0***
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - e.g. 1.0/0.0 = –1.0/–0.0 = +∞, 1.0/–0.0 = –∞
- *Case 2*: ***exp = 111…1, frac ≠ 000…0***
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - e.g. sqrt(–1), ∞ – ∞, ∞ * 0.0, …

Tiny FP Example (1)

- 8-bit floating point representation
  - The sign bit is in the most significant bit
  - The next four bits are the ***exp***, with a bias of 7
  - The last three bits are the ***frac***

- Same general form as IEEE format
  - Normalized, denormalized
  - Representation of 0, NaN, infinity

Tiny FP Example (2)

- Values related to the exponent (***Bias*** = 7)

Description	Exp	exp	E = Exp - Bias	2^E
Denormalized	0	0000	-6	1/64
Normalized	1	0001	-6	1/64
	2	0010	-5	1/32
	3	0011	-4	1/16
	4	0100	-3	1/8
	5	0101	-2	1/4
	6	0110	-1	1/2
	7	0111	0	1
	8	1000	1	2
	9	1001	2	4
	10	1010	3	8
	11	1011	4	16
	12	1100	5	32
	13	1101	6	64
	14	1110	7	128
inf, NaN	15	1111	-	-

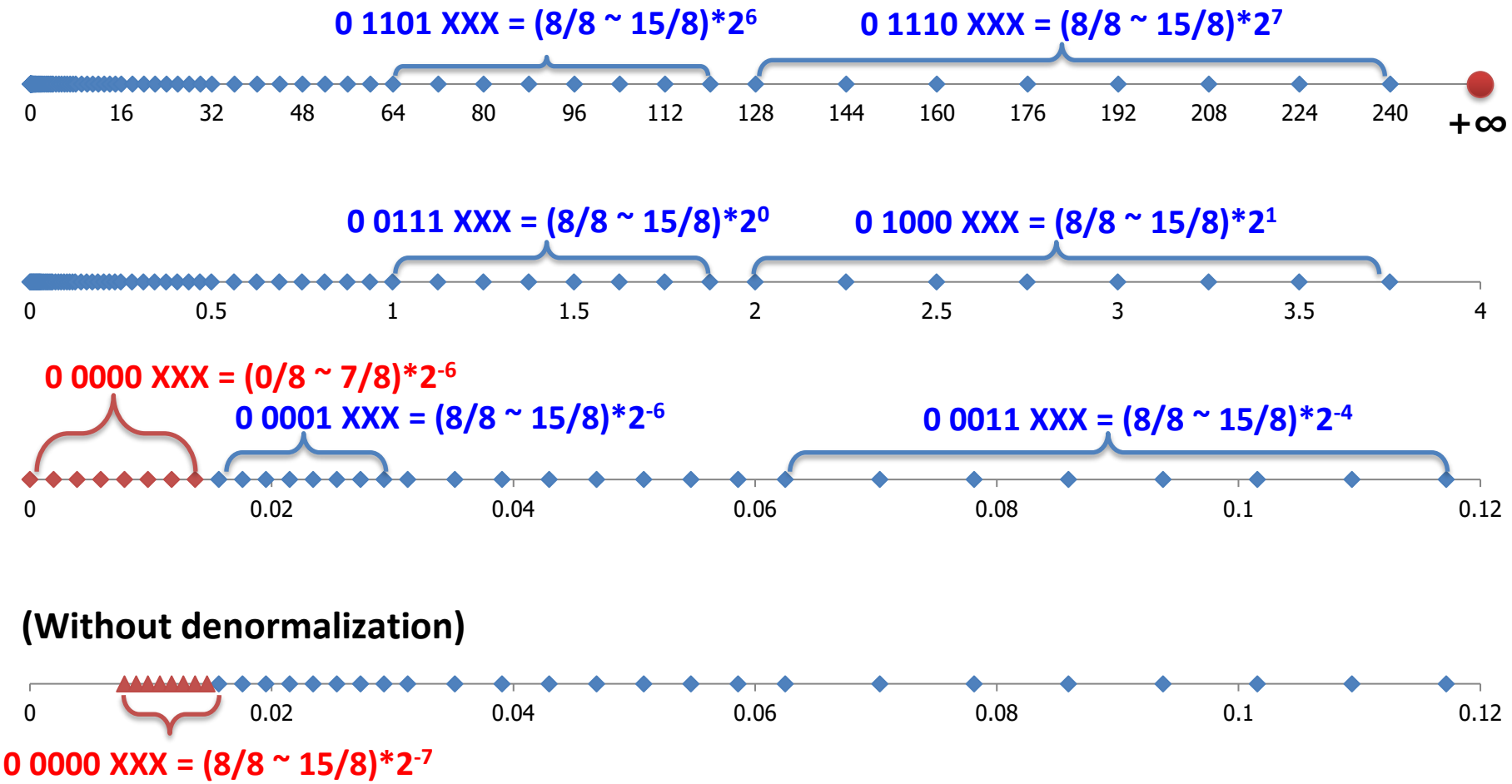
Tiny FP Example (3)

- Dynamic range

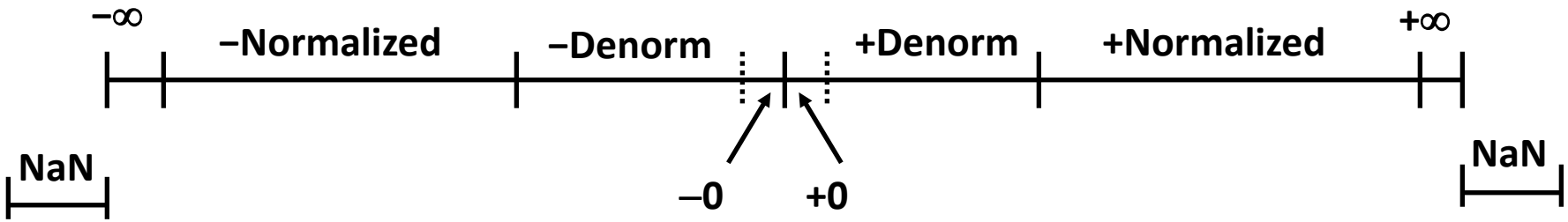
Description	Bit representation	e	E	f	M	V
Zero	0 0000 000	0	-6	0	0	0
Smallest pos.	0 0000 001	0	-6	1/8	1/8	1/512
	0 0000 010	0	-6	2/8	2/8	2/512
	0 0000 011	0	-6	3/8	3/8	3/512
	0 0000 110	0	-6	6/8	6/8	6/512
Largest denorm.	0 0000 111	0	-6	7/8	7/8	7/512
Smallest norm.	0 0001 000	1	-6	0	8/8	8/512
	0 0001 001	1	-6	1/8	9/8	9/512
	0 0110 110	6	-1	6/8	14/8	14/16
	0 0110 111	6	-1	7/8	15/8	15/16
One	0 0111 000	7	0	0	8/8	1
	0 0111 001	7	0	1/8	9/8	9/8
	0 0111 010	7	0	2/8	10/8	10/8
	0 1110 110	14	7	6/8	14/8	224
Largest norm.	0 1110 111	14	7	7/8	15/8	240
Infinity	0 1111 000	-	-	-	-	+∞

Tiny FP Example (4)

- Encoded values (nonnegative numbers only)

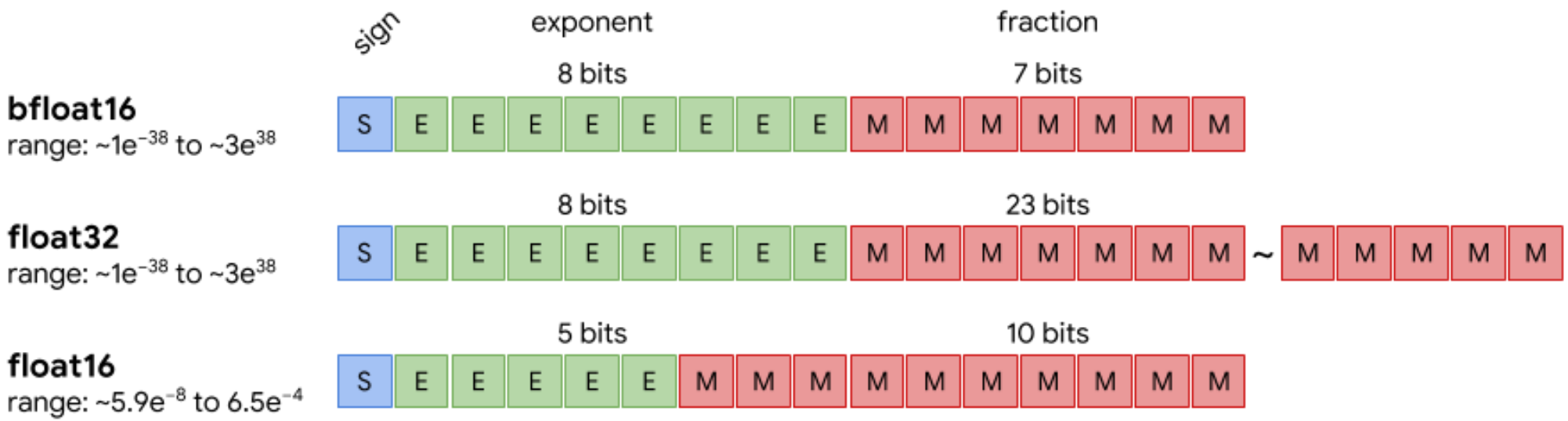


Interesting Numbers



Description	exp	frac	Numeric Value
Zero	000 … 00	000 … 00	0.0
Smallest Positive denormalized	000 … 00	000 … 01	Single: $2^{-23} \times 2^{-126} \approx 1.4 \times 10^{-45}$ Double: $2^{-52} \times 2^{-1022} \approx 4.9 \times 10^{-324}$
Largest Denormalized	000 … 00	111 … 11	Single: $(1.0 - \varepsilon) \times 2^{-126} \approx 1.18 \times 10^{-38}$ Double: $(1.0 - \varepsilon) \times 2^{-1022} \approx 2.2 \times 10^{-308}$
Smallest Positive Normalized	000 … 01	000 … 00	Single: 1.0×2^{-126}, Double: 1.0×2^{-1022} (Just larger than largest denormalized)
One	011 … 11	000 … 00	1.0
Largest Normalized	111 … 10	111 … 11	Single: $(2.0 - \varepsilon) \times 2^{127} \approx 3.4 \times 10^{38}$ Double: $(2.0 - \varepsilon) \times 2^{1023} \approx 1.8 \times 10^{308}$

IEEE FP16 vs. Google Bfloat16



- Google bfloat16
  - Introduced by Google in 2018 for TPUs (Supported by Intel NPUs too)
  - Same dynamic range as FP32
  - Smaller mantissa reduces power and physical silicon area

Manipulating Floating Points

Chap. 3.5, 3.9 – 3.10

Special Properties

- FP zero same as integer zero
  - All bits = 0

- Can (almost) use unsigned integer comparison
  - Must first compare sign bits
  - Must consider –0 = 0
  - NaNs problematic
    - Will be greater than any other values
  - Otherwise OK
    - Denormalized vs. normalized
    - Normalized vs. Infinity

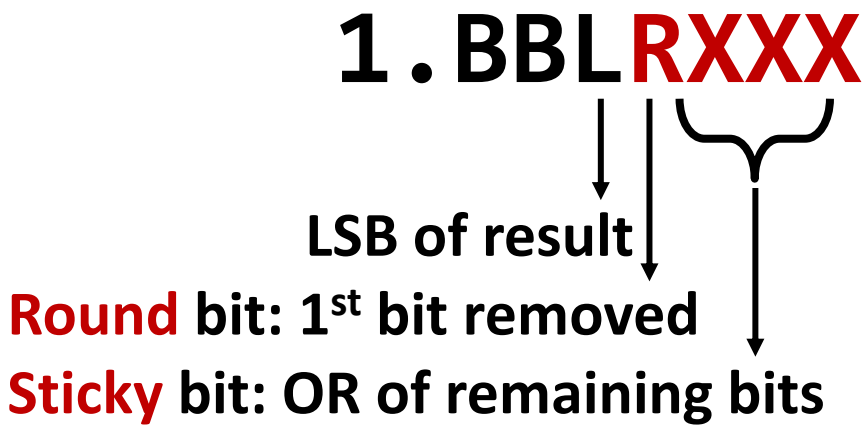
Rounding

- For a given value *x*, finding the "closest" matching value *x'* that can be represented in the FP format
- IEEE 754 defines four rounding modes
  - Round-to-even avoids statistical bias by rounding upward or downward so that the least significant digit is even

Rounding modes	$1.40	$1.60	$1.50	$2.50	$–1.50
Round-toward-zero	$1	$1	$1	$2	$–1
Round-down (–∞)	$1	$1	$1	$2	$–2
Round-up (+∞)	$2	$2	$2	$3	$–1
Round-to-even (default) or Round-to-nearest	$1	$2	$2	$2	$–2

Round-to-Even

- Round up conditions
  - R = 1, S = 1 → > 0.5
  - L = 1, R = 1, S = 0
    → Round to even



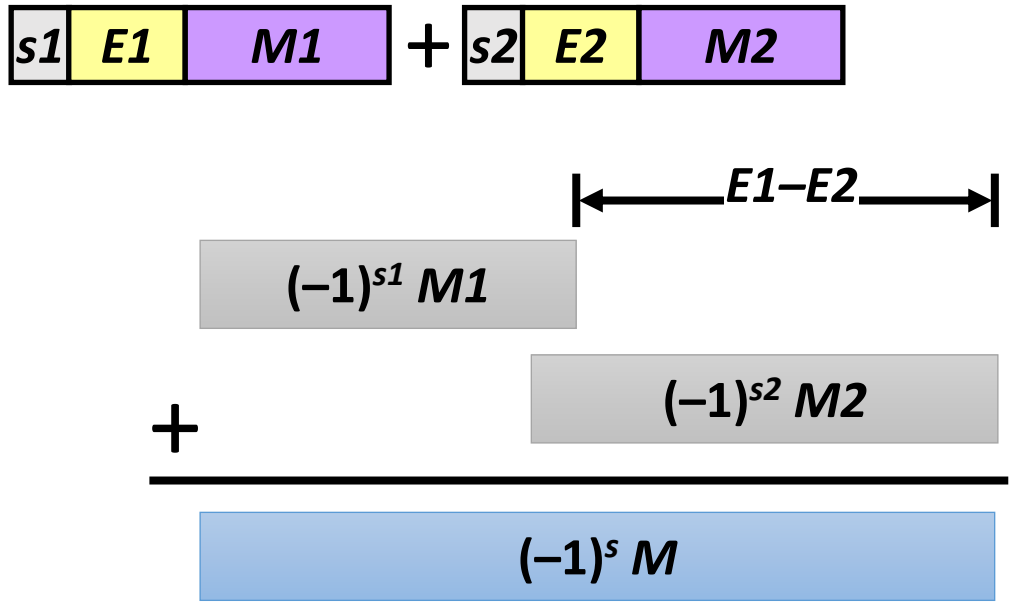
Value	Fraction	LRS	Up?	Rounded
128	1.0000000 (x2^7)	000	No	1.000
13	1.1010000 (x2^3)	100	No	1.101
17	1.0001000 (x2^4)	010	No	1.000
19	1.0011000 (x2^4)	110	Yes	1.010
138	1.0001010 (x2^7)	011	Yes	1.001
63	1.1111100 (x2^5)	111	Yes	10.000

FP Addition

- Adding two numbers:

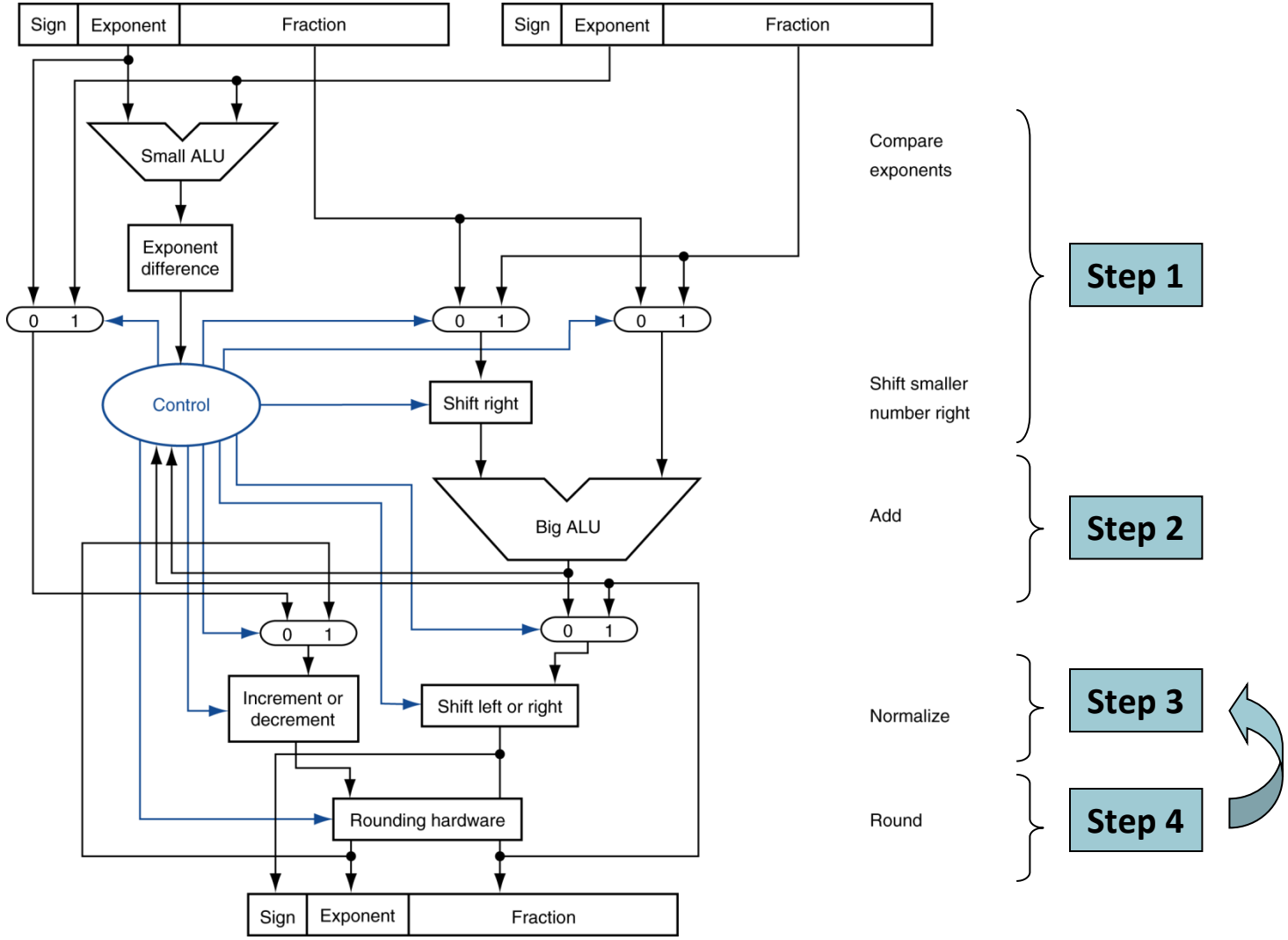
  (Assume *E1* > *E2*)

  1. Align binary points
     - Shift number with smallest exponent
     - Shift right *M2* by *E1* – *E2*
  2. Add significands
     - Result: Sign *s*, Significand *M*, Exponent *E* (= *E1*)
  3. Normalize result & check for over/underflow
     - if (*M* ≥ 2), shift *M* right, increment *E*
     - if (*M* < 1), shift *M* left *k* positions, decrement *E* by *k*
  4. Round *M* and renormalize if necessary



FP Adder Hardware

- Much more complex than integer adder



FP Multiplication

- Multiplying two numbers:

1. Add exponents
   - $E = E1 + E2$
2. Multiply significands
   - $M = M1 \times M2$
3. Normalize result & check for over/underflow
   check for overflow (E out of range?)
   - if ($M \geq 2$), shift M right, increment E
   - if ($M < 1$), shift M left k positions, decrement E by k
4. Round M and renormalize if necessary
5. Determine sign
   - $s = s1 \wedge s2$

FP Arithmetic Hardware

- FP multiplier is of similar complexity to FP adder
  - But uses a multiplier for significands instead of an adder
- FP arithmetic hardware usually does
  - Addition, subtraction, multiplication, division, reciprocal, square-root, …
  - FP ↔ integer conversion
- Completing an operation in one clock cycle would take too long
  - Much longer than integer operations
  - Slower clock would penalize all instructions
- Operations usually take several cycles
  - Can be pipelined

Floating Points in C

- C guarantees two levels
  - `float` (single precision) vs. `double` (double precision)

- Conversions
  - `double` or `float` → `int`
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN (Generally sets to TMin)
  - `int` → `double`
    - Exact conversion, as long as int has ≤ 53 bit word size
  - `int` → `float`
    - Will round according to rounding mode

FP Example 1

```
#include <stdio.h>

int main ()
{
      int n = 123456789;
      int nf, ng;
      float f;
      double g;

      f = (float) n;
      g = (double) n;
      nf = (int) f;
      ng = (int) g;
      printf ("nf=%d ng=%d\n", nf, ng);
}
```

FP Example 2

```
#include <stdio.h>

int main ()
{
    double  d;

    d = 1.0 + 0.1 + 0.1 + 0.1 + 0.1 + 0.1
            + 0.1 + 0.1 + 0.1 + 0.1 + 0.1;

    printf ("d = %.20f\n", d);
}
```

FP Example 3

```
#include <stdio.h>

int main ()
{
    float f1 = (3.14 + 1e20) – 1e20;
    float f2 = 3.14 + (1e20 – 1e20);

    printf (“f1 = %f, f2 = %f\n”, f1, f2);
}
```

Ariane 5

- Ariane 5 tragedy (June 4, 1996)
  - Exploded 37 seconds after liftoff
  - Satellites worth $500 million
- Why?
  - Computed horizontal velocity as floating-point number
  - Converted to 16-bit integer
    - Careful analysis of Ariane 4 trajectory proved 16-bit is enough
  - Reused a module from 10-year-old software
    - Overflowed for Ariane 5
    - No precise specification for the software

Byte Ordering

Chap. 2.3, 2.9

Data Types in C

C Data Type	Typical 32-bit	Typical 64-bit	x86-64
char	1	1	1
short	2	2	2
int	4	4	4
long	4	8	8
long long	8	8	8
float	4	4	4
double	8	8	8
long double	-	-	10/16
pointer	4	8	8

Byte Ordering

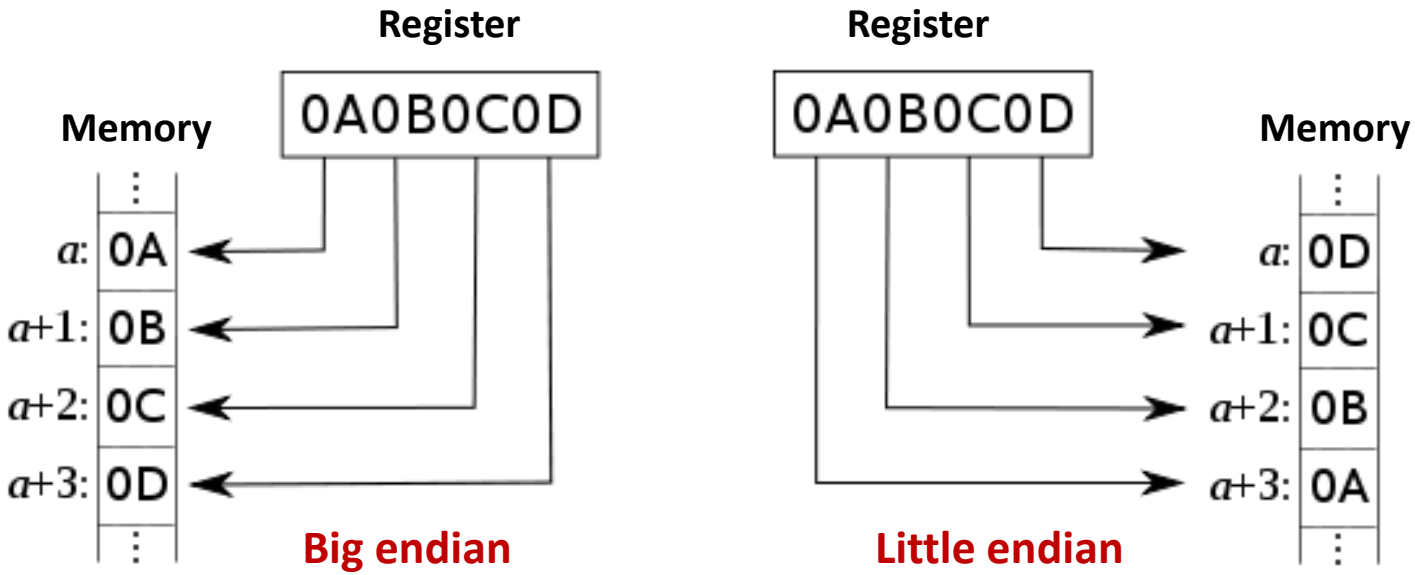
- How are the bytes within a multi-byte word ordered in memory?

- Conventions
  - Big endian: Sun, PowerPC Mac, Internet
  - Little endian: Intel x86, ARM running Android & iOS, RISC-V

- Note:
  - Alpha and PowerPC can run in either mode, with the byte ordering convention determined when the chip is powered up
  - Problem when the binary data is communicated over a network between different machines

Big vs. Little Endian

- Big endian
  - Least significant byte has highest address
- Little endian
  - Least significant byte has lowest address



Example

- What is the output of this program?
  - Solaris/SPARC: ?
  - Linux/x86-64: ?

```
#include   <stdio.h>

union {
    int             i;
    unsigned char   c[4];
} u;

int main ()  {
    u.i = 0x12345678;
    printf ("%x %x %x %x\n",
            u.c[0], u.c[1], u.c[2], u.c[3]);
}
```

Character Sets

- ASCII: 128 characters
  - 95 graphic, 33 control

ASCII Code Chart

	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	NUL	SOH	STX	ETX	EOT	ENQ	ACK	BEL	BS	HT	LF	VT	FF	CR	SO	SI
1	DLE	DC1	DC2	DC3	DC4	NAK	SYN	ETB	CAN	EM	SUB	ESC	FS	GS	RS	US
2		!	"	#	$	%	&	'	(	)	*	+	,	-	.	/
3	0	1	2	3	4	5	6	7	8	9	:	;	<	=	>	?
4	@	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	P	Q	R	S	T	U	V	W	X	Y	Z	[	\	]	^	_
6	`	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
7	p	q	r	s	t	u	v	w	x	y	z	{	|	}	~	DEL

- Latin-1: 256 characters
  - ASCII + 96 more graphic characters

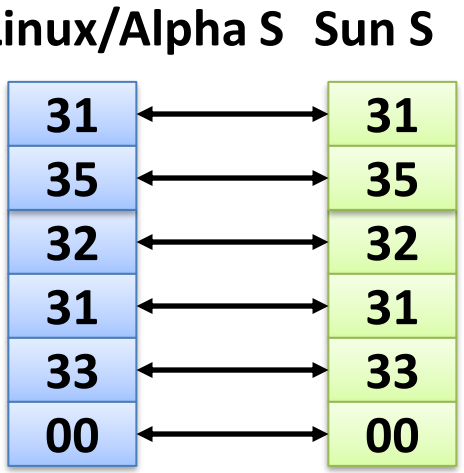
- Unicode: 32-bit character set
  - Used in Java, C++ wide characters, …
  - Most of the world's alphabets, plus symbols
  - UTF-8, UTF-16: variable-length encodings

Source: https://medium.com/lift-letters/ascii-characters-9ea3d9c68144

Representing Strings

- Strings in C
  - Represented by array of characters
  - Each character encoded in ASCII format
    - Standard 7-bit encoding of character set
    - Character '0' has code 0x30
    - Digit *i* has code 0x30 + *I*
  - String should be null-terminated
    - Final character = 0x00
- Compatibility
  - Byte ordering not an issue

char S[6] = "15213";



Summary

- Floating points important for scientific code (and machine learning)

- Computer representations of numbers
  - Finite range and precision
  - Overflow/underflow
  - Floating points violate associativity / distributivity
  - Different machines follow different byte ordering

- Bits have no inherent meaning
  - Interpretation depends on the instructions applied