Floating Points

Jin-Soo Kim
(jinsoo.kim@snu.ac.kr)

Systems Software &
Architecture Lab.

Seoul National University

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The Problem

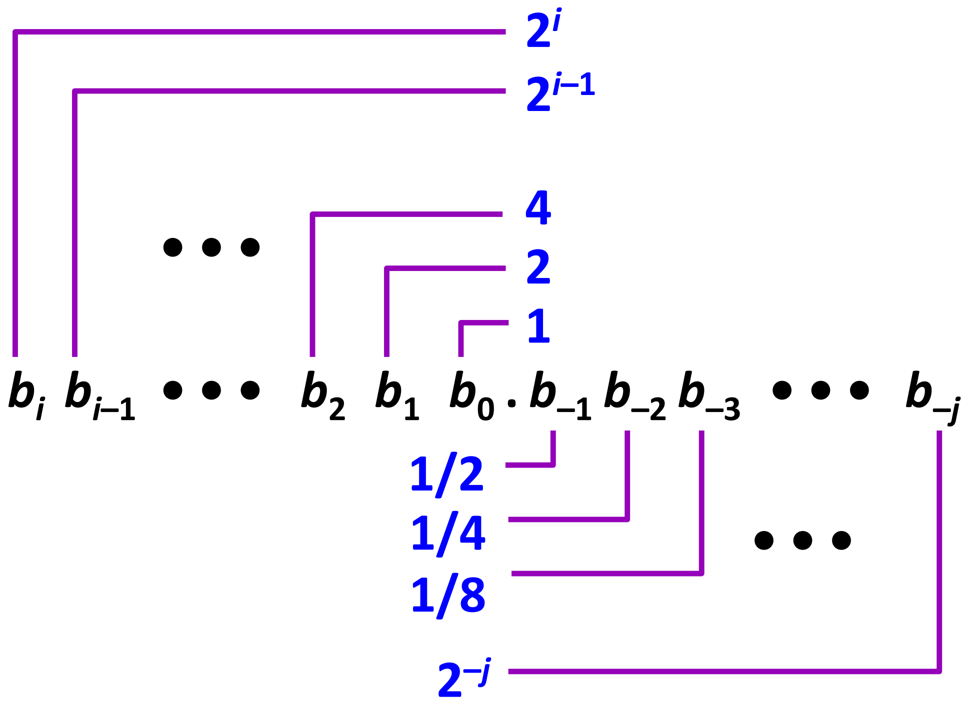
- How to represent fractional values with finite number of bits?
  - 0.1
  - 0.612
  - 3.14159265358979323846264338327950288...

- Wide ranges of numbers
  - 1 Light-Year = 9,460,730,472,580.8 km
  - The radius of a hydrogen atom: 0.000000000025 m

Fractional Binary Numbers (1)

- Representation
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number:

$$\sum_{k=-j}^{i} b_k \cdot 2^k$$



Fractional Binary Numbers (2)

- Examples:

Value	Representation
5-3/4	101.11_2
2-7/8	10.111_2
63/64	0.111111_2

- Observations
  - Divide by 2 by shifting right
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111..._2$ just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Fractional Binary Numbers (3)

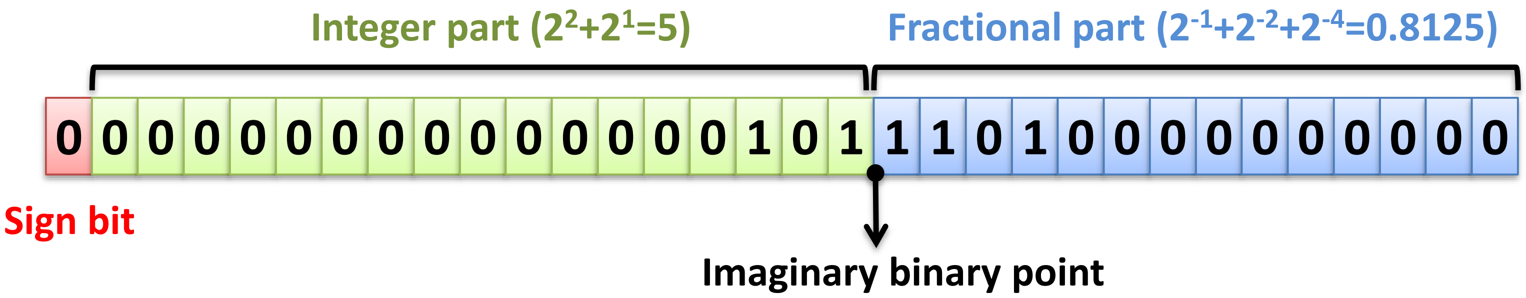
- Representable numbers
  - Can only exactly represent numbers of the form $x\ /\ 2^k$
  - Other numbers have repeating bit representations

Value	Representation
1/3	$0.0101010101[01]..._2$
1/5	$0.001100110011[0011]..._2$
1/10	$0.0001100110011[0011]..._2$

Fixed Points

Fixed-Point Representation (1)

- ***p.q*** Fixed-point representation
  - Use the rightmost ***q*** bits of an integer as representing a fraction
  - Example: 17.14 fixed-point representation
    - 1 bit for sign bit
    - 17 bits for the integer part
    - 14 bits for the fractional part
    - An integer ***x*** represents the real number $x / 2^{14}$
    - Maximum value: $(2^{31} - 1) / 2^{14} \fallingdotseq 131071.999$



Fixed-Point Representation (2)

- Properties
  - Convert ***n*** to fixed point: ***n*** * ***f*** (= ***n*** << ***q***)
  - Add ***x*** and ***y***: ***x*** + ***y***

	0	0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1	1 0 0 0 0 0 0 0 0 0 0 0 0 0 0	13.5
+	0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1	1 1 0 0 0 0 0 0 0 0 0 0 0 0 0	1.75
	0	0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1	0 1 0 0 0 0 0 0 0 0 0 0 0 0 0	15.25

  - Subtract ***y*** from ***x***: ***x*** – ***y***
  - Add ***x*** and ***n***: ***x*** + ***n*** * ***f***
  - Multiply ***x*** by ***n***: ***x*** * ***n***
  - Divide ***x*** by ***n***: ***x*** / ***n***

_x_, _y_: fixed-point number
_n_: integer
_f_ = 1 << _q_

Fixed-Point Representation (3)

- Pros
  - Simple
  - Can use integer arithmetic to manipulate
  - No floating-point hardware needed
  - Used in many low-cost embedded processors or DSPs (digital signal processors)

- Cons
  - Cannot represent wide ranges of numbers

Representing Floating Points

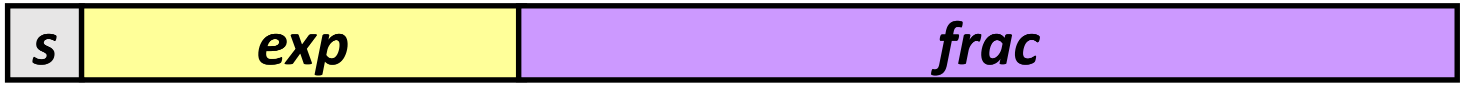
Representing Floating Points

- IEEE standard 754
  - Established in 1985 as uniform standard for floating-point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs
  - William Kahan, a primary architect of IEEE 754, won the Turing Award in 1989
  - Driven by numerical concerns
    - Nice standards for rounding, overflow, underflow
    - Hard to make go fast
    - Numerical analysts predominated over hardware types in defining standard

FP Representation

- Numerical form: $-1^s \times M \times 2^E$
  - Sign bit ***s*** determines whether number is negative or positive
  - Significand ***M*** normally a fractional value in range [1.0, 2.0)
  - Exponent ***E*** weights value by power of two

- Encoding



  - MSB is sign bit ***s***
  - ***exp*** field encodes ***E*** (Exponent)
  - ***frac*** field encodes ***M*** (Mantissa)

FP Precisions

- Single precision
  - 8 ***exp*** bits, 23 ***frac*** bits (32 bits total)
- Double precision
  - 11 ***exp*** bits, 52 ***frac*** bits (64 bits total)
- Extended precision
  - 15 ***exp*** bits, 63 ***frac*** bits
  - Only found in Intel-compatible machines
  - Stored in 80 bits (1 bit wasted)

Normalized Values

- Condition: ***exp*** $\neq$ ***000…0*** and ***exp*** $\neq$ ***111…1***
- Exponent coded as a biased value
  - ***E*** = ***Exp*** – ***Bias***
  - ***Exp***: unsigned value denoted by ***exp***
  - ***Bias***: Bias value ($=2^{k-1}-1$, where *k* is the number of ***exp*** bits)
    - Single precision (*k*=8): 127 (***Exp***: 1..254, ***E***: -126..127)
    - Double precision (*k*=11): 1023 (***Exp***: 1..2046, ***E***: -1022..1023)
- Significand coded with implied leading 1
  - ***M*** = $1.xxx…x_2$
    - Minimum when ***frac*** = 000…0 (***M*** = 1.0)
    - Maximum when ***frac*** = 111…1 (***M*** = 2.0 – ε)
  - Get extra leading bit for "free"

Normalized Values: Example

- `float f = 2003.0;`
  - 2003_{10} = 11111010011_2 = $1.1111010011_2 \times 2^{10}$
- Significand
  - ***M*** = $1.\underline{1111010011}_2$
  - ***frac*** = $\underline{1111010011}0000000000000_2$
- Exponent
  - ***E*** = 10
  - ***Exp*** = ***E*** + ***Bias*** = 10 + 127 = 137 = 10001001_2

Hex:	**4**	**4**	**F**	**A**	**6**	**0**	**0**	**0**
Binary:	**0100**	**0100**	**1111**	**1010**	**0110**	**0000**	**0000**	**0000**
137:	**100**	**0100**	**1**					
2003:			***1*111**	**1010**	**0110**			

Denormalized Values

- Condition: ***exp* = *000…0***
- Value
  - Exponent value ***E*** = 1 – ***Bias***
  - Significand value ***M*** = $0.xxx…x_2$ (no implied leading 1)
- *Case 1*: ***exp* = *000…0***, ***frac* = *000…0***
  - Represents value 0.0
  - Note that there are distinct values +0 and -0
- *Case 2*: ***exp* = *000…0***, ***frac* ≠ *000…0***
  - Numbers very close to 0.0
  - "Gradual underflow": possible numeric values are spaced evenly near 0.0

Special Values

- Condition: ***exp* = *111…1***
- *Case 1*: ***exp* = *111…1*, *frac* = *000…0***
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - e.g. 1.0/0.0 = –1.0/–0.0 = +∞, 1.0/–0.0 = –∞
- *Case 2*: ***exp* = *111…1*, *frac* ≠ *000…0***
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - e.g. sqrt(–1), ∞ – ∞, ∞ * 0.0, …

Tiny FP Example (1)

- 8-bit floating point representation
  - The sign bit is in the most significant bit
  - The next four bits are the ***exp***, with a bias of 7
  - The last three bits are the ***frac***

- Same general form as IEEE format
  - Normalized, denormalized
  - Representation of 0, NaN, infinity

Tiny FP Example (2)

- Values related to the exponent (***Bias*** = 7)

Description	Exp	exp	E = Exp - Bias	2^E
Denormalized	0	0000	-6	1/64
Normalized	1	0001	-6	1/64
	2	0010	-5	1/32
	3	0011	-4	1/16
	4	0100	-3	1/8
	5	0101	-2	1/4
	6	0110	-1	1/2
	7	0111	0	1
	8	1000	1	2
	9	1001	2	4
	10	1010	3	8
	11	1011	4	16
	12	1100	5	32
	13	1101	6	64
	14	1110	7	128
inf, NaN	15	1111	-	-

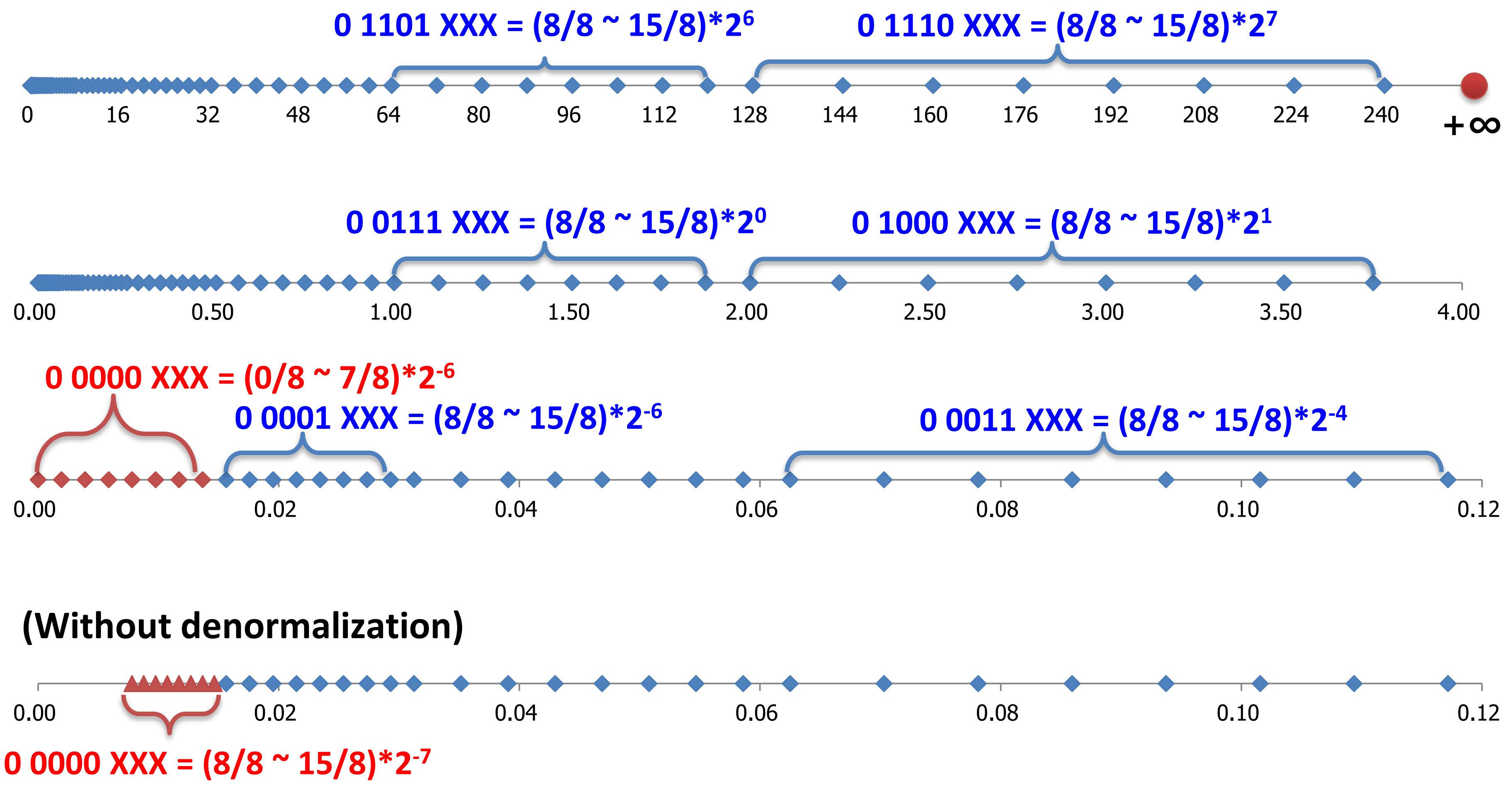
Tiny FP Example (3)

- Dynamic range

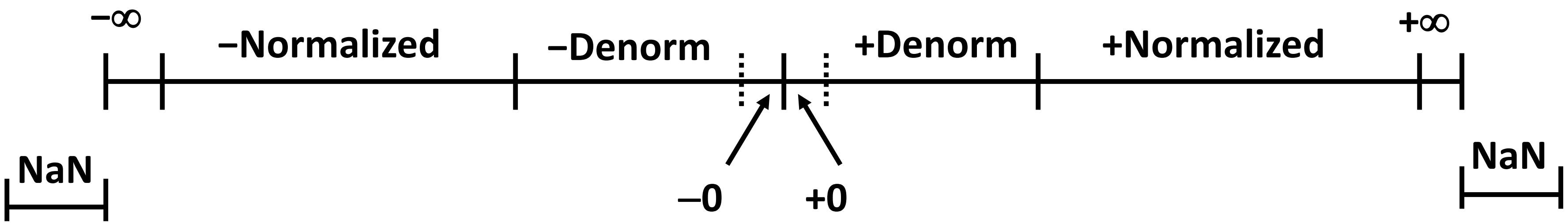
Description	Bit representation	e	E	f	M	V
Zero	0 0000 000	0	-6	0	0	0
Smallest pos.	0 0000 001	0	-6	1/8	1/8	1/512
	0 0000 010	0	-6	2/8	2/8	2/512
	0 0000 011	0	-6	3/8	3/8	3/512
	0 0000 110	0	-6	6/8	6/8	6/512
Largest denorm.	0 0000 111	0	-6	7/8	7/8	7/512
Smallest norm.	0 0001 000	1	-6	0	8/8	8/512
	0 0001 001	1	-6	1/8	9/8	9/512
	0 0110 110	6	-1	6/8	14/8	14/16
	0 0110 111	6	-1	7/8	15/8	15/16
One	0 0111 000	7	0	0	8/8	1
	0 0111 001	7	0	1/8	9/8	9/8
	0 0111 010	7	0	2/8	10/8	10/8
	0 1110 110	14	7	6/8	14/8	224
Largest norm.	0 1110 111	14	7	7/8	15/8	240
Infinity	0 1111 000	-	-	-	-	+∞

Tiny FP Example (4)

- Encoded values (nonnegative numbers only)



Interesting Numbers



Description	exp	frac	Numeric Value
Zero	000 … 00	000 … 00	0.0
Smallest Positive denormalized	000 … 00	000 … 01	Single: 2^{-23} X 2^{-126} ≈ 1.4 X 10^{-45} Double: 2^{-52} X 2^{-1022} ≈ 4.9 X 10^{-324}
Largest Denormalized	000 … 00	111 … 11	Single: (1.0 – ε) X 2^{-126} ≈ 1.18 X 10^{-38} Double: (1.0 – ε) X 2^{-1022} ≈ 2.2 X 10^{-308}
Smallest Positive Normalized	000 … 01	000 … 00	Single: 1.0 X 2^{-126}, Double: 1.0 X 2^{-1022} (Just larger than largest denormalized)
One	011 … 11	000 … 00	1.0
Largest Normalized	111 … 10	111 … 11	Single: (2.0 – ε) X 2^{127} ≈ 3.4 X 10^{38} Double: (2.0 – ε) X 2^{1023} ≈ 1.8 X 10^{308}

Manipulating Floating Points

Special Properties

- FP zero same as integer zero
  - All bits = 0

- Can (almost) use unsigned integer comparison
  - Must first compare sign bits
  - Must consider –0 = 0
  - NaNs problematic
    - Will be greater than any other values
  - Otherwise OK
    - Denormalized vs. normalized
    - Normalized vs. Infinity

Rounding

- For a given value *x*, finding the "closest" matching value *x'* that can be represented in the FP format
- IEEE 754 defines four rounding modes
  - Round-to-even avoids statistical bias by rounding upward or downward so that the least significant digit is even

Rounding modes	\$1.40	\$1.60	\$1.50	\$2.50	\$–1.50
Round-toward-zero	\$1	\$1	\$1	\$2	\$–1
Round-down (–∞)	\$1	\$1	\$1	\$2	\$–2
Round-up (+∞)	\$2	\$2	\$2	\$3	\$–1
Round-to-even (default) or Round-to-nearest	\$1	\$2	\$2	\$2	\$–2

Round-to-Even

- Round up conditions
  - R = 1, S = 1 → > 0.5
  - G = 1, R = 1, S = 0
    → Round to even

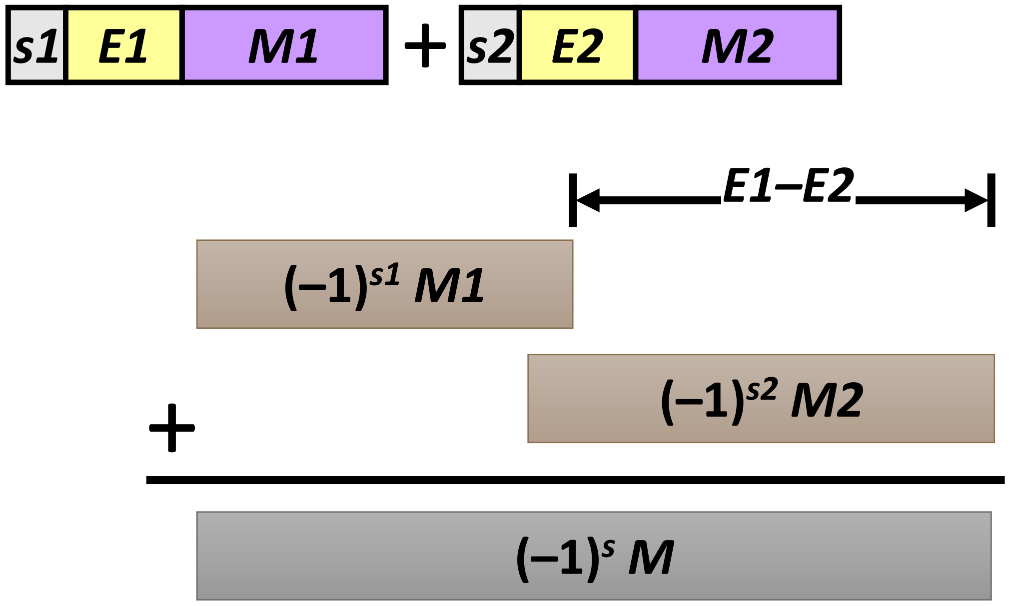
1.BBGRXXX

Guard bit: LSB of result
Round bit: 1st bit removed
Sticky bit: OR of remaining bits

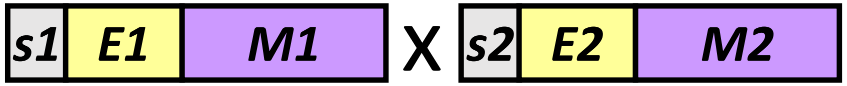
Value	Fraction	GRS	Up?	Rounded
128	1.0000000 (x2^7)	000	No	1.000
13	1.1010000 (x2^3)	100	No	1.101
17	1.0001000 (x2^4)	010	No	1.000
19	1.0011000 (x2^4)	110	Yes	1.010
138	1.0001010 (x2^7)	011	Yes	1.001
63	1.1111100 (x2^5)	111	Yes	10.000

FP Addition

- Adding two numbers:
  (Assume $E1 > E2$)
  - Align binary points
    - Shift right $M2$ by $E1 - E2$
  - Add significands
    - Result: Sign s, Significand M, Exponent E ($= E1$)
  - Normalize result
    - if ($M \geq 2$), shift M right, increment E
    - if ($M < 1$), shift M left k positions, decrement E by k
  - Check for overflow (E out of range?)
  - Round M and renormalize if necessary



FP Multiplication



- Multiplying two numbers:
  - Obtain exact result
    - Sign *s* = *s1* ^ *s2*
    - Significand *M* = *M1* x *M2*
    - Exponent *E* = *E1* + *E2*
    - The biggest chore is multiplying significands
  - Normalize result
    - if (*M* ≥ 2), shift *M* right, increment *E*
    - if (*M* < 1), shift *M* left *k* positions, decrement *E* by *k*
  - Check for overflow (*E* out of range?)
  - Round *M* and renormalize if necessary

Floating Points in C

- C guarantees two levels
  - **`float`** (single precision) vs. **`double`** (double precision)

- Conversions
  - **`double`** or **`float`** → **`int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN (Generally sets to TMin)
  - **`int`** → **`double`**
    - Exact conversion, as long as int has ≤ 53 bit word size
  - **`int`** → **`float`**
    - Will round according to rounding mode

```
unsigned int x = (0 << 31) | (0x7f << 23) | 0;
printf(“%f\n”, (float) x);
```

FP Example 1

```c
#include <stdio.h>

int main ()
{
        int n = 123456789;
        int nf, ng;
        float f;
        double g;

        f = (float) n;
        g = (double) n;
        nf = (int) f;
        ng = (int) g;
        printf (“nf=%d ng=%d\n”, nf, ng);
}
```

FP Example 2

```
#include <stdio.h>

int main ()
{
     double  d;

     d = 1.0 + 0.1 + 0.1 + 0.1 + 0.1 + 0.1
             + 0.1 + 0.1 + 0.1 + 0.1 + 0.1;

     printf ("d = %.20f\n", d);
}
```

FP Example 3

```c
#include <stdio.h>

int main ()
{
    float f1 = (3.14 + 1e20) – 1e20;
    float f2 = 3.14 + (1e20 – 1e20);

    printf ("f1 = %f, f2 = %f\n", f1, f2);
}
```

Ariane 5

- Ariane 5 tragedy (June 4, 1996)
  - Exploded 37 seconds after liftoff
  - Satellites worth $500 million
- Why?
  - Computed horizontal velocity as floating-point number
  - Converted to 16-bit integer
    - Careful analysis of Ariane 4 trajectory proved 16-bit is enough
  - Reused a module from 10-year-old software
    - Overflowed for Ariane 5
    - No precise specification for the software

Summary

- IEEE floating point has clear mathematical properties
- Represents numbers of form $\boldsymbol{M} \times \boldsymbol{2^E}$
- Can reason about operations independent of implementation
  - As if computed with perfect precision and then rounded
- Not the same as real arithmetic
  - Violates associativity / distributivity
  - Makes life difficult for compilers and serious numerical applications programmers